# Trees with Maximum $p$-Reinforcement Number*

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**Abstract**

Let $G=(V,E)$ be a graph and $p$ a positive integer. The $p$-domination number $\gamma_p(G)$ is the minimum cardinality of a set $D\subseteq V$ with $|N_G(x)\cap D|\geq p$ for all $x\in V\setminus D$. The $p$-reinforcement number $r_p(G)$ is the smallest number of edges whose addition to $G$ results in a graph $G'$ with $\gamma_p(G')<\gamma_p(G)$. Recently, it was proved by Lu et al. that $r_p(T)\leq p+1$ for a tree $T$ and $p\geq 2$. In this paper, we characterize all trees attaining this upper bound for $p\geq 3$.

**Keywords:** $p$-domination, $p$-reinforcement number, trees

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## 1 Induction

For notation and graph-theoretical terminology not defined here we follow [5, 19]. Let $G=(V,E)$ be a finite, undirected and simple graph with vertex-set $V=V(G)$ and edge-set $E=E(G)$. For a vertex $x\in V$, its *open neighborhood*, *closed neighborhood* and *degree* are respectively $N_G(x)=\{y\in V: xy\in E\}$, $N_G[x]=N_G(x)\cup\{x\}$ and

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$deg_G(x) = |N_G(x)|$. A vertex of degree one is called an *endvertex* or a *leaf* and its neighbor is called a *stem*. We denote the set of leaves of $G$ by $L(G)$.

For $S \subseteq V(G)$, the subgraph induced by $S$ (resp. $V(G) \setminus S$) is denoted by $G[S]$ (resp. $G - S$). The complement $G^c$ of $G$ is the simple graph whose vertex-set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of $G$. For $B \subseteq E(G^c)$, we use $G + B$ to denote the subgraph with vertex-set $V(G)$ and edge-set $E(G) \cup B$. For convenience, for vertex $v \in V(G)$, edge $xy \in E(G^c)$ and subgraph $H \subseteq G$, we write $G - \{v\}$, $G + \{xy\}$ and $G - V(H)$ for $G - v$, $G + xy$ and $G - H$, respectively.

Let $T$ be a tree and $xy \in E(T)$. Use the notation $T_y$ to denote the component of $T - x$ containing $y$. To simplify notation, we still use $T_y$ to denote $V(T_y)$. If $T$ is a rooted tree, then, for every $x \in V(T)$, we let $C(x)$ and $D(x)$ denote the sets of children and descendants, respectively, of $x$, and define $D[x] = D(x) \cup \{x\}$.

Let $p$ be a positive integer. A subset $D \subseteq V(G)$ is a *$p$-dominating set* of $G$ if every vertex not in $D$ has at least $p$ neighbors in $D$. The *$p$-domination number* $\gamma_p(G)$ is the minimum cardinality of a $p$-dominating set of $G$. A $p$-dominating set with cardinality $\gamma_p(G)$ is called a $\gamma_p(G)$-set. The *$p$-reinforcement number* $r_p(G)$ is the smallest number of edges of $G^c$ that have to be added to $G$ in order to reduce $\gamma_p(G)$, that is

$$r_p(G) = \min\{|B| \ : \ B \subseteq E(G^c) \ \text{ with } \ \gamma_p(G + B) < \gamma_p(G)\}.$$

By convention $r_p(G) = 0$ if $\gamma_p(G) \leq p$. Clearly, the 1-domination and 1-reinforcement numbers are the well-known domination and reinforcement numbers, respectively.

The $p$-domination was introduced by Fink and Jacobson [11] and has been well studied in graph theory (see, for example, [1, 2, 6, 8, 9, 10, 12]). Very recently, Chellali et al. [4] gave an excellent survey on this topics. The $p$-reinforcement number introduced by Lu et al. [17] is a parameter for measuring the vulnerability of the $p$-domination, is also a natural extension of the classical reinforcement number which was introduced by Kok and Mynhardt [16] and has been studied by a number of authors including [7, 14, 13, 15, 20]. Motivated by the works of these authors, Lu et al. [17] give an original study on the $p$-reinforcement for any $p \geq 1$. For a graph $G$ and $p \geq 1$, they found a method to determine $r_p(G)$ in terms of $\gamma_p(G)$, show that the decision problem on $r_p(G)$ is NP-hard and established some upper bounds.

To be surprising, for a tree $T$, the upper bounds of $r_p(T)$ have distinct difference between $p = 1$ and $p \geq 2$. For $p = 1$, Blair et al. [3] proved that $r_1(T) \leq \frac{1}{2}|V(T)|$ and this bound is sharp. However, the following theorem implies that no result in terms of $|V(T)|$ exists for $p \geq 2$.

**Theorem 1.1** ([17]) *$r_p(T) \leq p + 1$ for a tree $T$ and $p \geq 2$.*

In this paper, we continue to consider the $p$-reinforcement number of trees. We will focus on the structural properties of the extremal trees in Theorem 1.1, and character all extremal trees for $p \geq 3$ by a recursive construction.

The rest of this paper is organized as follows. In Section 2 we present some notations and known results. We show the structural properties of a tree $T$ with $r_p(T) = p+1$ for $p \geq 3$ in Sections 3, and then characterize such trees in Section 4.

## 2 Known Results

In this section, we will make the necessary preparations for proving the main results in Sections 3 and 4. Let $G = (V, E)$ be a graph and $p$ be a positive integer.

**Observation 2.1** *Every $p$-dominating set contains all vertices of degree at most $p-1$.*

For $X \subseteq V$, let $X^* = \{x \in V \setminus X : |N_G(x) \cap X| < p\}$, and define

$$\begin{aligned} \eta_p(x, X, G) &= \begin{cases} p - |N_G(x) \cap X| & \text{if } x \in X^*; \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } x \in V, & (2.1) \\ \eta_p(S, X, G) &= \sum_{x \in S} \eta_p(x, X, G) \quad \text{for } S \subseteq V, & (2.2) \\ \eta_p(G) &= \min\{\eta_p(V, X, G) : |X| < \gamma_p(G)\}. & (2.3) \end{aligned}$$

A subset $X \subseteq V$ is called an *$\eta_p(G)$-set* if $\eta_p(G) = \eta_p(V, X, G)$.

**Observation 2.2** *If $X$ is an $\eta_p(G)$-set, then $|X| = \gamma_p(G) - 1$.*

**Theorem 2.3** ([17]) *For a graph $G$ and $p \geq 1$, $r_p(G) = \eta_p(G)$ if $\gamma_p(G) \geq p+1$.*

**Corollary 2.1** ([17]) *Let $G$ be a graph and $p \geq 1$. For any $H \subseteq G$ with $\gamma_p(H) \geq p+1$ and $\gamma_p(G) \geq \gamma_p(H) + \gamma_p(G-H)$,*

$$r_p(G) \leq r_p(H).$$

Let $X \subseteq V$ and $x \in X$. A vertex $y \in V \setminus X$ is called a *$p$-private neighbor* of $x$ with respect to $X$ if $xy \in E$ and $|N_G(y) \cap X| = p$. The *$p$-private neighborhood* of $x$ with respect to $X$, denoted by $PN_p(x, X, G)$, is defined as the set of $p$-private neighbors of $x$ with respect to $X$. Let

$$\begin{aligned} \mu_p(x, X, G) &= |PN_p(x, X, G)| + \max\{0, p - |N_G(x) \cap X|\}, & (2.4) \\ \mu_p(X, G) &= \min\{\mu_p(x, X, G) : x \in X\}, & (2.5) \\ \mu_p(G) &= \min\{\mu_p(X, G) : \ X \text{ is a } \gamma_p(G)\text{-set}\}. & (2.6) \end{aligned}$$

**Theorem 2.4** ([17]) *For a graph $G$ and $p \geq 1$, $r_p(G) \leq \mu_p(G)$ with equality if $r_p(G) = 1$.*

## 3 Structural Theorems

In this section, we investigate some structural properties of trees with $r_p(T) = p + 1$. We need the following lemma which is given in [18].

**Lemma 3.1** ([18]) *Let $p \geq 2$ and $T$ be a tree with a $p$-dominating set $D$. Then $D$ is a unique $\gamma_p(T)$-set if and only if $D$ satisfies either $|N_T(x) \cap D| \leq p - 2$ or $|PN_p(x, D, T)| \geq 2$ for each $x \in D$ with degree at least $p$.*

**Theorem 3.1** *Let $p \geq 2$ and $T$ be a tree with $r_p(T) = p + 1$. For a $\gamma_p(T)$-set $D$,*

**(i)** *$PN_p(x, D, T) \neq \emptyset$ for each $x \in D$;*

**(ii)** *$D$ is a unique $\gamma_p(T)$-set.*

**Proof.** Let $x$ be a vertex in $D$. By the definitions of $\mu_p$ in (2.4)$\sim$(2.6) and Theorem 2.4, we have

$$\begin{aligned} |PN_p(x, D, T)| &= \mu_p(x, D, T) - \max\{0, p - |N_T(x) \cap D|\} \\ &\geq \mu_p(D, T) - p \\ &\geq \mu_p(T) - p \\ &\geq r_p(T) - p \\ &= 1. \end{aligned}$$

The first conclusion holds.

We now prove the second conclusion. It is obvious that $|D| = \gamma_p(T) \geq p + 1$ since $r_p(T) = p + 1 > 0$. If every vertex in $D$ has degree less than $p$, then $D$ is a unique $\gamma_p(T)$-set by Observation 2.1. Assume that $D$ has a vertex, say $x$, with degree at least $p$. By the definitions of $\mu_p$ in (2.4)$\sim$(2.6) and Theorem 2.4, we have

$$r_p(T) \leq \mu_p(T) \leq \mu_p(x, D, T) = |PN_p(x, D, T)| + \max\{0, p - |N_T(x) \cap D|\}. \quad (3.1)$$

If $|N_T(x) \cap D| \geq p - 1$, then $\max\{0, p - |N_T(x) \cap D|\} \leq 1$, and so we obtain from (3.1) that $|PN_p(x, D, T)| \geq r_p(T) - 1 = p \geq 2$. This fact implies that $D$ satisfies the conditions of Lemma 3.1. Thus $D$ is a unique $\gamma_p(T)$-set. The theorem follows. ∎

Through this paper, for any tree $T$ with a unique $\gamma_p(T)$-set, we denote the unique $\gamma_p(T)$-set by $\mathscr{D}_T$ for short.

**Theorem 3.2** *Let $p \geq 2$ and $T$ be a tree with $r_p(T) = p+1$. For edge $xy \in E(T)$ with $x \in \mathscr{D}_T$, let $T_y$ denote the component of $T - x$ containing $y$. Then*

**(i)** *If $y \in PN_p(x, \mathscr{D}_T, T)$, then either $T_y$ is a star $K_{1,p-1}$ with center $y$, or $r_p(T_y) = 1$ and $\mathscr{D}_T \cap T_y$ is an $\eta_p(T_y)$-set. Moreover, $\eta_p(T_y, S, T_y) \geq p - 1$ for $S \subseteq T_y$ with $|S| = |\mathscr{D}_T \cap T_y|$ and $y \in S$.*

**(ii)** *If $y \notin PN_p(x, \mathscr{D}_T, T)$, then $r_p(T_y) = p + 1$ and $\mathscr{D}_{T_y} = \mathscr{D}_T \cap T_y$.*

**Proof. (i)** Since $x \in \mathscr{D}_T$ and $y \in PN_p(x, \mathscr{D}_T, T)$, we have $y \notin \mathscr{D}_T$ and $|N_T(y) \cap \mathscr{D}_T| = p$. It follows that

$$|T_y| \geq |((N_T(y) \cap \mathscr{D}_T) \setminus \{x\}) \cup \{y\}| = p.$$

If $|T_y| = p$, then it is easy to see that $T_y = K_{1,p-1}$ with center $y$, and $\eta_p(T_y, S, T_y) = p-1$ for any $S \subseteq T_y$ with $|S| = |\mathscr{D}_T \cap T_y|$ and $y \in S$. Assume that $|T_y| \geq p+1$ in the following.

**Claim.** $\gamma_p(T_y) \geq p + 1$.

**Proof of Claim 3.** Suppose, to be contrary, that $\gamma_p(T_y) = p$, and let $D_y$ be a $\gamma_p(T_y)$-set. Then there is a vertex, say $z$, of $T_y$ not in $D_y$ since $|T_y| \geq p+1$. To $p$-dominate $z$, $|N_{T_y}(z) \cap D_y| \geq p = |D_y| \geq 2$. From the fact that two vertices in a tree have at most one common neighbor, we know that $z$ is a unique vertex of $T_y$ not in $D_y$. Thus $T_y = K_{1,p}$ and $z = y$. Hence, by Observation 2.1, we can obtain a contradiction as follows:

$$p = |N_T(y) \cap \mathscr{D}_T| = |N_T(y)| = |(T_y \setminus \{y\}) \cup \{x\}| = 1 + p.$$

The claim holds. □

**Claim.** $\eta_p(T_y, S, T_y) \geq \begin{cases} 1 & \text{if } y \notin S; \\ p-1 & \text{if } y \in S, \end{cases}$ for any $S \subseteq T_y$ with $|S| = |\mathscr{D}_T \cap T_y|$.

**Proof of Claim 3.** Let $S$ be a subset of $T_y$ such that $|S| = |\mathscr{D}_T \cap T_y|$. We complete the proof of Lemma 3 by distinguishing the following three cases.

If $S = \mathscr{D}_T \cap T_y$, then it is obvious that $\eta_p(T_y, S, T_y) = 1$ by $y \in PN_p(x, \mathscr{D}_T, T)$.

If $S \neq \mathscr{D}_T \cap T_y$ and $y \notin S$, then let $D' = (\mathscr{D}_T \setminus T_y) \cup S$. Clearly, $|D'| = |\mathscr{D}_T| = \gamma_p(T)$ and $D' \neq \mathscr{D}_T$. Since $\mathscr{D}_T$ is a unique $\gamma_p(T)$-set, $D'$ is not a $p$-dominating set of $T$. Thus, by $x \in \mathscr{D}_T$ and $x \in D'$, we have

$$\eta_p(T_y, S, T_y) \geq \eta_p(T_y, D', T) = \eta_p(V(T), D', T) \geq 1.$$

If $y \in S$, then, for any $u \in N_T(y)$, we use $T_u$ to denote the component of $T - y$ containing $u$. Since $y \in S \setminus \mathscr{D}_T$ and $|S| = |\mathscr{D}_T \cap T_y|$, we have

$$\begin{aligned} \sum_{u \in N_T(y) \setminus \{x\}} |S \cap T_u| = |S \setminus \{y\}| &= |S| - 1 \\ &= |\mathscr{D}_T \cap T_y| - 1 = \sum_{u \in N_T(y) \setminus \{x\}} |\mathscr{D}_T \cap T_u| - 1. \end{aligned}$$

Thus, there is some $v \in N_T(y) \setminus \{x\}$ such that $|S \cap T_v| \leq |\mathscr{D}_T \cap T_v| - 1$. Let

$$D'' = (\mathscr{D}_T \setminus T_v) \cup (S \cap T_v).$$

Then

$$\begin{aligned}|D''| &\leq |\mathscr{D}_T \setminus T_v| + |T_v \cap S| \\ &\leq |\mathscr{D}_T \setminus T_v| + (|\mathscr{D}_T \cap T_v| - 1) \\ &= |\mathscr{D}_T| - 1 \\ &= \gamma_p(T) - 1,\end{aligned}$$

and $\eta_p(y, D'', T) \leq 1$ since $|N_T(y) \cap D''| \geq |N_T(y) \cap \mathscr{D}_T - \{v\}| \geq |N_T(y) \cap \mathscr{D}_T| - 1 = p - 1$. By Theorem 2.3 and the definitions of $\eta_p$ in (2.1)$\sim$(2.3), we have

$$\begin{aligned}p + 1 = r_p(T) = \eta_p(T) &\leq \eta_p(V(T), D'', T) \\ &= \eta_p(T_v, D'', T) + \eta_p(y, D'', T) \\ &\leq \eta_p(T_v, D'', T) + 1 \\ &= \eta_p(T_v, T_v \cap S, T_y) + 1.\end{aligned}$$

Thus, $\eta_p(T_v, S \cap T_v, T_y) \geq p$, and so

$$\begin{aligned}\eta_p(T_y, S, T_y) &\geq \eta_p(T_v, S, T_y) \\ &= \eta_p(T_v, S \cap T_v, T_y) - 1 \\ &\geq p - 1.\end{aligned}$$

The proof of Claim 3 is complete. $\Box$

Claim 3 implies that the second conclusion in **(i)** holds. We now show that $r_p(T_y) = 1$ and $\mathscr{D}_T \cap T_y$ is an $\eta_p(T_y)$-set. By $p \geq 2$, it is easily seen from Claim 3 that $\eta_p(T_y, S, T_y) \geq 1$ for $S \subseteq T_y$ with $|S| = |\mathscr{D}_T \cap T_y|$. So $\gamma_p(T_y) \geq |\mathscr{D}_T \cap T_y| + 1$, further, we have $\gamma_p(T_y) = |\mathscr{D}_T \cap T_y| + 1$ since $(\mathscr{D}_T \cap T_y) \cup \{y\}$ is a $p$-dominating set of $T_y$. By Claim 3 and its proof, $\eta_p(T_y) = \eta_p(T_y, \mathscr{D}_p \cap T_y, T_y) = 1$ and $\mathscr{D}_T \cap T_y$ is an $\eta_p(T_y)$-set. Hence $r_p(T_y) = \eta_p(T_y) = 1$ by Theorem 2.3 and Claim 3.

The proof of **(i)** is complete.

**(ii)** By the hypothesis of $y \notin PN_p(x, \mathscr{D}_T, T)$, $\mathscr{D}_T \setminus T_y$ and $\mathscr{D}_T \cap T_y$ are $p$-dominating sets of $T - T_y$ and $T_y$, respectively. Thus, we have

$$|\mathscr{D}_T \setminus T_y| \geq \gamma_p(T - T_y) \ \text{ and } \ |\mathscr{D}_T \cap T_y| \geq \gamma_p(T_y). \tag{3.2}$$

Note that the union of a $\gamma_p(T - T_y)$-set and a $\gamma_p(T_y)$-set is a $p$-dominating set of $T$. Thus, we have

$$\gamma_p(T - T_y) + \gamma_p(T_y) = \gamma_p(T). \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}\gamma_p(T - T_y) + \gamma_p(T_y) = \gamma_p(T) = |\mathscr{D}_T| &= |\mathscr{D}_T \setminus T_y| + |\mathscr{D}_T \cap T_y| \\ &\geq \gamma_p(T - T_y) + \gamma_p(T_y),\end{aligned}$$

which yields that $|\mathscr{D}_T \cap T_y| = \gamma_p(T_y)$, that is, $\mathscr{D}_T \cap T_y$ is a $\gamma_p(T_y)$-set.

To the end, by **(ii)** of Theorem 3.1, it is sufficient to prove $r_p(T_y) = p + 1$.

Arbitrary take $u \in \mathscr{D}_T \cap T_y$. Since $r_p(T) = p+1$, we have $PN_p(u, \mathscr{D}_T, T) \neq \emptyset$ by **(i)** of Theorem 3.1. Let $z \in PN_p(u, \mathscr{D}_T, T)$. Clearly, $z \in T_y$ and $z \neq y$ since $u \in T_y$ and $y \notin PN_p(x, \mathscr{D}_T, T)$. So

$$|N_{T_y}(z) \cap (\mathscr{D}_T \cap T_y)| = |N_T(z) \cap \mathscr{D}_T| = p.$$

It follows that

$$\gamma_p(T_y) = |\mathscr{D}_T \cap T_y| \geq |N_{T_y}(z) \cap (\mathscr{D}_T \cap T_y)| = p.$$

Furthermore, we can show that $\gamma_p(T_y) \geq p+1$. To be contrary, assume that $\gamma_p(T_y) = p$. Then $|\mathscr{D}_T \cap T_y| = p$ and $z$ is a unique vertex of $T_y$ not in $\mathscr{D}_T \cap T_y$ since two vertices in a tree have at most one common neighbor. It follows that $T_y = K_{1,p}$ with center $z$ and $\mathscr{D}_T \cap T_y = T_y \setminus \{z\}$, and so $y \in \mathscr{D}_T$ and $y$ is a leaf of $T_y$. This implies that $N_T(y) = \{x, z\}$ and by (2.4),

$$\begin{aligned} \mu_p(y, \mathscr{D}_T, T) &= |PN_p(y, \mathscr{D}_T, T)| + \max\{0, p - |N_T(y) \cap \mathscr{D}_T|\} \\ &= 1 + (p-1) = p. \end{aligned}$$

By Theorem 2.4 and the definitions of $\mu_p$ in (2.4)$\sim$(2.6), we have

$$r_p(T) \leq \mu_p(T) \leq \mu_p(y, \mathscr{D}_T, T) = p,$$

which contradicts to the hypothesis of $r_p(T) = p+1$. Thus $\gamma_p(T_y) \geq p+1$.

Let $X$ be an $\eta_p(T_y)$-set and $Y = X \cup (\mathscr{D}_T \setminus T_y)$. Then $|X| = \gamma_p(T_y) - 1$ and

$$|Y| = (\gamma_p(T_y) - 1) + (\gamma_p(T) - \gamma_p(T_y)) = \gamma_p(T) - 1.$$

By Theorem 2.3, we have

$$\begin{aligned} r_p(T_y) &= \eta_p(T_y, X, T_y) \\ &\geq \eta_p(T_y, Y, T) = \eta_p(V(T), Y, T) \\ &\geq \eta_p(T) = r_p(T) = p+1. \end{aligned}$$

Combining this with Theorem 1.1, we have $r_p(T_y) = p+1$, and so **(ii)** is true.

The theorem follows. ∎

**Lemma 3.2** *Assume that $p \geq 3$ and $T$ be a tree with $r_p(T) = p+1$. Let $x \in \mathscr{D}_T$ and $X \subseteq V(T-x)$ with $|X| < \gamma_p(T)$. If $\mu_p(x, \mathscr{D}_T, T) \geq p+2$ and $\eta_p(V(T), X, T) = p+1$, then $|X \cap T_y| = |\mathscr{D}_T \cap T_y|$ for all $y \in N_T(x)$, where $T_y$ is the component of $T-x$ containing $y$.*

**Proof.** Let $N_T(x) = \{x_1, \ldots, x_d\}$, where $d = deg_T(x)$, and let $T_i$ be the component of $T-x$ containing $x_i$. Combining $x \in \mathscr{D}_T \setminus X$ with $|X| < \gamma_p(T)$, we have

$$\sum_{j=1}^{d} |X \cap T_j| = |X| \leq \gamma_p(G) - 1 = |\mathscr{D}_T \setminus \{x\}| = \sum_{j=1}^{d} |\mathscr{D}_T \cap T_j|.$$

We only need to show that $|X \cap T_j| = |\mathscr{D}_T \cap T_j|$ for any $j \in \{1, \ldots, d\}$. Suppose, to the contrary, that there exists some $i \in \{1, \ldots, d\}$ such that

$$|X \cap T_i| < |\mathscr{D}_T \cap T_i|.$$

Our aim is to deduce a contradiction.

We first give the following two claims. Claim 3 can be easily obtained from Theorem 3.2,.

**Claim.** For any $j \in \{1, \ldots, d\}$,

$$|\mathscr{D}_T \cap T_j| = \begin{cases} \gamma_p(T_j) - 1 & \text{if } \ x_j \in PN_p(x, \mathscr{D}_T, T); \\ \gamma_p(T_j) & \text{if } \ x_j \notin PN_p(x, \mathscr{D}_T, T). \end{cases}$$

**Claim.** For any $j \in \{1, \ldots, d\}$,

$$|X \cap T_j| \geq \begin{cases} \gamma_p(T_j) - 2 & \text{if } \ j = i; \\ \gamma_p(T_j) & \text{if } \ j \neq i. \end{cases}$$

**Proof of Claim 3.** Let $D = (X \cap T_i) \cup (\mathscr{D}_T \setminus T_i)$. Clearly,

$$|D| = |X \cap T_i| + |\mathscr{D}_T \setminus T_i| < |\mathscr{D}_T \cap T_i| + |\mathscr{D}_T \setminus T_i| = |\mathscr{D}_T| = \gamma_p(T).$$

Thus $\eta_p(V(T), D, T) \geq \eta_p(T)$ by (2.3). From $x \in \mathscr{D}_T \setminus T_i \subseteq D$, we know that $D$ is a $p$-dominating set of $T - T_i$, and so $\eta_p(T_i, D, T) = \eta_p(V(T), D, T)$. Note that $\gamma_p(T) \geq p+1$ by $r_p(T) = p+1$. Together with the hypothesis of Lemma 3.2, (2.1)$\sim$(2.3) and Theorem 2.3, we have

$$\begin{array}{rcl} p+1 & = & \eta_p(V(T), X, T) \\ & \geq & \eta_p(V(T), X, T) - \displaystyle\sum_{j \neq i} \eta_p(T_j, X, T) - \eta_p(x, X, T) \\ & = & \eta_p(T_i, X, T) \\ & \geq & \eta_p(T_i, D, T) = \eta_p(V(T), D, T) \\ & \geq & \eta_p(T) = r_p(T) \\ & = & p+1, \end{array} \tag{3.4}$$

which implies that all the equalities in (3.4) hold. In particular,

$$\eta_p(x, X, T) = \eta_p(T_j, X, T) = 0, \ \text{ for } j \neq i,$$

which means that

$$|N_T(x) \cap X| \geq p \ \text{ and} \tag{3.5}$$
$$|N_{T_j}(u) \cap X| \geq p, \ \text{ for } j \neq i \text{ and } u \in T_j \setminus X. \tag{3.6}$$

It follows from (3.6) and $x \notin X$ that $X \cap T_j$ (for $j \neq i$) is a $p$-dominating set of $T_j$, and so

$$|X \cap T_j| \geq \gamma_p(T_j) \ \text{ for } j \neq i.$$

Also since all the equalities in (3.4) hold, we have $\eta_p(V(T), D, T) = r_p(T)$, which means that $D$ is an $\eta_p(T)$-set. By Observation 2.2,

$$|D| = \gamma_p(T) - 1 = |\mathscr{D}_T| - 1. \tag{3.7}$$

Since $x \in \mathscr{D}_T$ and $x_i \in N_T(x)$, $(\mathscr{D}_T \cap T_i) \cup \{x_i\}$ is a $p$-dominating set of $T_i$, and so

$$|\mathscr{D}_T \cap T_i| \geq \gamma_p(T_i) - 1. \tag{3.8}$$

It follows from the definition of $D$, (3.7) and (3.8) that

$$\begin{array}{rcl}|X \cap T_i| & = & |D| - |\mathscr{D}_T \setminus T_i| \\ & = & (|\mathscr{D}_T| - 1) - |\mathscr{D}_T \setminus T_i| \\ & = & |\mathscr{D}_T \setminus (\mathscr{D}_T \setminus T_i)| - 1 \\ & = & |\mathscr{D}_T \cap T_i| - 1 \\ & \geq & \gamma_p(T_i) - 2.\end{array}$$

Claim 3 holds. $\Box$

Together with Claims 3 and 3, we have that

$$\begin{array}{rcl}\gamma_p(T) & \geq & |X| + 1 = \sum\limits_{j \neq i} |X \cap T_j| + |X \cap T_i| + 1 \\ & \geq & \sum\limits_{j \neq i} \gamma_p(T_j) + (\gamma_p(T_i) - 2) + 1 \\ & = & \sum\limits_{i=1}^{d} \gamma_p(T_j) - 1 \\ & = & \sum\limits_{x_j \in PN_p(x, \mathscr{D}_T, T)} \gamma_p(T_j) + \sum\limits_{x_j \notin PN_p(x, \mathscr{D}_T, T)} \gamma_p(T_j) - 1 \\ & = & \sum\limits_{x_j \in PN_p(x, \mathscr{D}_T, T)} (|\mathscr{D}_T \cap T_j| + 1) + \sum\limits_{x_j \notin PN_p(x, \mathscr{D}_T, T)} (|\mathscr{D}_T \cap T_j|) - 1 \\ & = & |PN_p(x, \mathscr{D}_T, T)| + \sum\limits_{i=1}^{d} |\mathscr{D}_T \cap T_j| - 1 \\ & = & |PN_p(x, \mathscr{D}_T, T)| + (|\mathscr{D}_T| - 1) - 1 \quad (\text{since } x \in \mathscr{D}_T) \\ & = & |PN_p(x, \mathscr{D}_T, T)| + \gamma_p(T) - 2,\end{array} \tag{3.9}$$

which yields

$$|PN_p(x, \mathscr{D}_T, T)| \leq 2. \tag{3.10}$$

It follows from our hypothesis of $\mu_p(x, \mathscr{D}_T, T) \geq p + 2$ and (3.10) that

$$\begin{array}{rcl}p + 2 & \leq & \mu_p(x, \mathscr{D}_T, T) \\ & = & |PN_p(x, \mathscr{D}_T, T)| + \max\{0, p - |N_T(x) \cap \mathscr{D}_T|\} \\ & \leq & 2 + \max\{0, p - |N_T(x) \cap \mathscr{D}_T|\} \\ & \leq & p + 2,\end{array}$$

which implies that

$$|N_T(x) \cap \mathscr{D}_T| = 0 \quad \text{and} \tag{3.11}$$
$$|PN_p(x, \mathscr{D}_T, T)| = 2. \tag{3.12}$$

It follows from (3.12) that all equalities in (3.9) hold, and so, by Claim 3,

$$|X \cap T_j| = \begin{cases} \gamma_p(T_i) - 2 & \text{if } j = i; \\ \gamma_p(T_j) & \text{if } j \neq i. \end{cases} \tag{3.13}$$

We claim that $x_i \in PN_p(x, \mathscr{D}_T, T)$. Assume, to be contrary, that $x \notin PN_p(x, \mathscr{D}_T, T)$, then $r_p(T_i) = p+1$ by **(ii)** of Theorem 3.2. Since $\eta_p(T_i, X \cap T_i, T_i) = \eta_p(T_i, X, T) = p+1$ by $x \notin X$ and (3.4), $X \cap T_i$ is an $\eta_p(T_i)$-set, and so $|X \cap T_i| = \gamma_p(T_i) - 1$ by Observation 2.2, which contradicts with (3.13). The claim holds, and so if $x_j \notin PN_p(x, \mathscr{D}_T, T)$ then $j \neq i$.

For $j \neq i$, it can be easily seen from (3.13) that $X \cap T_j$ is a $\gamma_p(T_j)$-set since $X \cap T_j$ is a $p$-dominating set of $T_j$ by (3.6) and $x \notin X$. On the other hand, for any $x_j \notin PN_p(x, \mathscr{D}_T, T)$, $\mathscr{D}_T \cap T_j$ is a unique $\gamma_p(T_j)$-set by **(ii)** of Theorem 3.2. Hence

$$X \cap T_j = \mathscr{D}_T \cap T_j, \quad \text{for } x_j \notin PN_p(x, \mathscr{D}_T, T). \tag{3.14}$$

Together with (3.11), (3.12) and (3.14), we have $|N_T(x) \cap X| \leq |PN_p(x, \mathscr{D}_T, T)| = 2$, which contradicts to (3.5) since $p \geq 3$. The lemma follows. ∎

**Remark 1** *The conclusion of Lemma 3.2 may not be valid for* $p = 2$. *An counterexample is showed in Figure 1.*

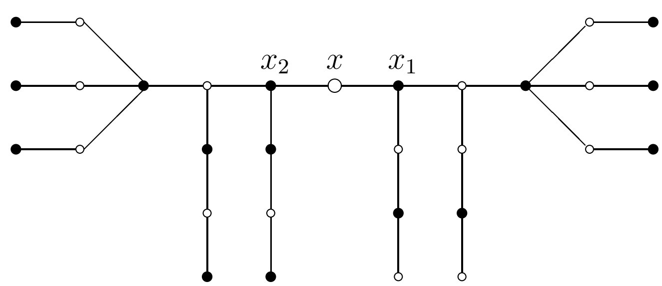


Figure 1: A trees $T$ with $\gamma_2(T) = 17$ and $r_2(T) = 3$. The vertex $x$ satisfies $\mu_2(x, \mathscr{D}_T, T) = 4$ and the set $X$ of black vertices in $T$ has $|X| = 16 < \gamma_2(T)$ and $\eta_2(V(T), X, T) = 3$. However, $|X \cap T_1| = 7 \neq 8 = |\mathscr{D}_T \cap T_1|$ and $|X \cap T_2| = 9 \neq 8 = |\mathscr{D}_T \cap T_2|$, where $T_1$ and $T_2$ are two components of $T - x$ containing $x_1$ and $x_2$, respectively.

**Theorem 3.3** *Let* $p \geq 3$, $T$ *be a tree with* $r_p(T) = p+1$ *and* $x \in \mathscr{D}_T$. *If* $\mu_p(x, \mathscr{D}_T, T) \geq p+2$, *then* $\eta_p(V(T), X, T) \geq p+2$ *for any* $X \subseteq V(T - x)$ *with* $|X| < \gamma_p(T)$.

**Proof.** Suppose, to the contrary, that $\eta_p(V(T), X, T) \leq p+1$ for some subset $X \subseteq V(T-x)$ with $|X| < \gamma_p(T)$. Since $\eta_p(V(T), X, T) \geq r_p(T) = p+1$ by Theorem 2.3, we have

$$\eta_p(V(T), X, T) = p+1. \tag{3.15}$$

Let $T_y$ denote the component of $T-x$ containing $y$ for any $y \in N_T(x)$. We partition $N_T(x)$ into six subsets $N_i$ $(1 \leq i \leq 6)$ such that $N_T(x) = \bigcup_{i=1}^{6} N_i$, where

$$\left\{ \begin{array}{ll} N_1 = PN_p(x, \mathscr{D}_T, T) \cap X, & N_2 = PN_p(x, \mathscr{D}_T, T) - X, \\ N_3 = (N_T(x) \cap X) \cap \mathscr{D}_T, & N_4 = (N_T(x) - X) \cap \mathscr{D}_T, \\ N_5 = (N_T(x) \cap X) - N_1 - N_3, & N_6 = (N_T(x) - X) - N_2 - N_4. \end{array} \right\}$$

Clearly,

$$|N_1| + |N_2| = |PN_p(x, \mathscr{D}_T, T)| \tag{3.16}$$

and, by $x \notin X$ and the definition of $\eta_p$ in (2.1),

$$\eta_p(x, X, T) = \max\{0, p - |N_1| - |N_3| - |N_5|\}. \tag{3.17}$$

For any $y \in N_1 \cup N_2 \cup N_5$, we can obtain $|X \cap T_y| = |\mathscr{D}_T \cap T_y|$ from (3.15) and Lemma 3.2. If $y \in N_1 \cup N_2$, then $y \in PN_p(x, \mathscr{D}_T, T)$ and, by **(i)** of Theorem 3.2, we have

$$\eta_p(T_y, X, T) = \eta_p(T_y, X \cap T_y, T_y) \geq \left\{ \begin{array}{ll} p-1 & \text{if } y \in N_1; \\ 1 & \text{if } y \in N_2. \end{array} \right. \tag{3.18}$$

If $y \in N_5$, then $y \notin PN_p(x, \mathscr{D}_T, T)$ and $y \in X \setminus \mathscr{D}_T$, which implies that $X \cap T_y \neq \mathscr{D}_T \cap T_y$. By $|X \cap T_y| = |\mathscr{D}_T \cap T_y|$ and Theorem 3.2 **(ii)**, $X \cap T_y$ is not a $p$-dominating set of $T_y$, and so

$$\eta_p(T_y, X, T) = \eta_p(T_y, X \cap T_y, T_y) \geq 1. \tag{3.19}$$

Therefore, we can derive from (3.16)$\sim$(3.19) that

$$\begin{array}{rcl} \eta_p(V(T), X, T) & = & \eta_p(x, X, T) + \displaystyle\sum_{y \in N_T(x)} \eta_p(T_y, X, T) \\ & \geq & \max\{0, p - |N_1| - |N_3| - |N_5|\} + \displaystyle\sum_{y \in N_1 \cup N_2 \cup N_5} \eta_p(T_y, X, T) \\ & \geq & \max\{0, p - |N_3|\} - |N_1| - |N_5| + (p-1)|N_1| + |N_2| + |N_5| \\ & = & |PN_p(x, \mathscr{D}_T, T)| + \max\{0, p - |N_3|\} + (p-3)|N_1| \\ & \geq & |PN_p(x, \mathscr{D}_T, T)| + \max\{0, p - |N_T(x) \cap \mathscr{D}_T|\} + 0 \\ & = & \mu_p(x, \mathscr{D}_T, T) \\ & \geq & p+2, \end{array}$$

which contradicts with (3.15). ∎

# 4 A constructive Characterization

In this section, we will give a constructive characterization of trees with $r_p(T) = p+1$ for $p \geq 3$. To the end, assume that $p \geq 3$.

Let $t \geq 2$ be an integer. The unique stem of a star $K_{1,t}$ is called the *center* of $K_{1,t}$. A *spider* $S_t$ is a tree obtained by attaching one leaf at each endvertex of $K_{1,t}$. Two important trees $F_{p-1}$ and $F_{t,p-1}$ in our construction are shown in Figure 2, where $F_{p-1}$ (resp. $F_{t,p-1}$) is obtained by attaching $p-1$ leaves at every endvertex of a path $P_3$ (resp. a spider $S_t$).

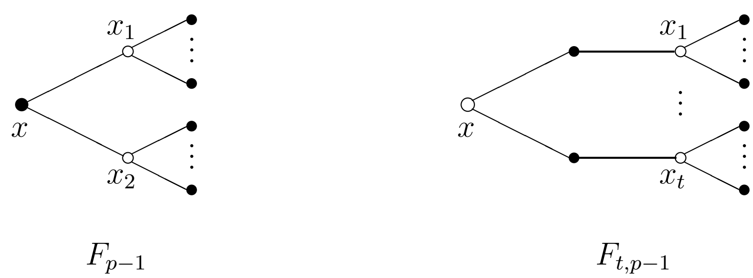


Figure 2: Trees $F_{p-1}$ and $F_{t,p-1}$, where $t \geq p$ and each $x_i$ has $p-1$ leaves.

In Figure 2, we call $x$ the *center* of $F_{p-1}$ and $F_{t,p-1}$. Clearly, the set of black vertices in $F_{p-1}$ (resp. $F_{t,p-1}$) is a unique $\gamma_p(F_{p-1})$-set (resp. $\gamma_p(F_{t,p-1})$-set). By Theorem 2.3, the following lemma is straightforward by computing $\eta_p$.

**Lemma 4.1** *If $p \geq 3$, then $r_p(F_{p-1}) = p+1$ and $r_p(F_{t,p-1}) = p+1$ for any $t \geq p$.*

Let $G$ and $H$ be two graphs. For $x \in V(G)$ and $y \in V(H)$, the notation $G \uplus_{xy} H$ is a graph consisting of $G$ and $H$ with an extra edge $xy$. Let $T_0 = K_{1,p}$ be a star and $A(T_0) = L(K_{1,p})$. We define a family $\mathscr{T}_p$ of trees as follows:

$$\mathscr{T}_p = \{T : T \text{ is obtained from a sequence } T_0, T_1, \ldots, T_k\ (k \geq 1) \text{ of trees, where } T = T_k \text{ and } T_{i+1}\ (0 \leq i \leq k-1) \text{ is obtained from } T_i \text{ by one of the operations listed below}\}.$$

Four operations on a tree $T_i$:

$\mathscr{O}_1$: $T_{i+1} = K_{1,p-1} \uplus_{xy} T_i$, where $x$ is the center of $K_{1,p-1}$ and $y \in A(T_i)$.
Let $A(T_{i+1}) = L(K_{1,p-1}) \cup A(T_i)$.

$\mathscr{O}_2$: $T_{i+1} = K_{1,p} \uplus_{xy} T_i$, where $x$ is the center of $K_{1,p}$ and $y \notin A(T_i)$.
Let $A(T_{i+1}) = L(K_{1,p}) \cup A(T_i)$.

$\mathscr{O}_3$: $T_{i+1} = F_{p-1} \uplus_{xy} T_i$, where $x$ is the center of $F_{p-1}$ and $y \in A(T_i)$ satisfying $|PN_p(y, A(T_i), T_i)| \geq \min\{p+1, |N_{T_i}(y) \cap A(T_i)| + 2\}$.
Let $A(T_{i+1}) = \mathscr{D}_{F_{p-1}} \cup A(T_i)$.

$\mathscr{O}_4$: $T_{i+1} = F_{t,p-1} \uplus_{xy} T_i$, where $t \geq p$, $x$ is the center of $F_{t,p-1}$ and $y \in T_i$.
Let $A(T_{i+1}) = \mathscr{D}_{F_{t,p-1}} \cup A(T_i)$.

**Lemma 4.2** *Let $p \geq 3$ be an integer. If $T \in \mathscr{T}_p$, then $A(T)$ is a unique $\gamma_p(T)$-set and $r_p(T) = p+1$.*

**Proof.** Since $T \in \mathscr{T}_p$, there is an integer $k \geq 1$ such that $T = T_k$ obtained from a sequence $T_0, T_1, \ldots, T_{k-1}, T_k$ of trees, where $T_0 = K_{1,p}$ and $T_{i+1}$ $(0 \leq i \leq k-1)$ can be obtained from $T_i$ by some $\mathscr{O}_j$ $(j \in \{1,2,3,4\})$. We complete the proof of Lemma 4.2 by induction on $k$.

For $k = 1$, it is trivial that $A(T_1)$ is a unique $\gamma_p(T_1)$-set and $r_p(T_1) = \eta_p(T_1) = p+1$ by straightly computing $\eta_p(T_1)$. This establishes the induction base.

Let $k \geq 2$. Assume that $A(T_i)$ is a unique $\gamma_p(T_i)$-set and $r_p(T_i) = p+1$ for $1 \leq i \leq k-1$. Basing on this assumption, we will prove that $A(T_k)$ is a unique $\gamma_p(T_k)$-set and $r_p(T_k) = p+1$.

Since $T_k$ is obtained from $T_{k-1}$ by $\mathscr{O}_j$ for some $j$, we denote $G_j = T_k - V(T_{k-1})$ and so $T_k = G_j \uplus_{xy} T_{k-1}$, where $x$ is the center of $G_j$ and $y$ satisfies the condition of $\mathscr{O}_j$.

We first show that $A(T_k)$ is a unique $\gamma_p(T_k)$-set. Applying the induction on $T_{k-1}$, $A(T_{k-1})$ is a unique $\gamma_p(T_{k-1})$-set. For $v \in A(T_{k-1})$ with degree at least $p$, it follows from Lemma 3.1 that either $|N_{T_{k-1}}(x) \cap A(T_{k-1})| \leq p-2$ or $|PN_p(v, A(T_{k-1}), T_{k-1})| \geq 2$. Hence, we can easily derive from the definition of $A(T_k)$ that $A(T_k)$ is a $p$-dominating set of $T_k$, moreover, satisfies the conditions of Lemma 3.1. By Lemma 3.1, $A(T_k)$ is a unique $\gamma_p(T_k)$-set

Next we only need to prove that $r_p(T_k) \geq p+1$ by Theorem 1.1. Assume, to be contrary, that $r_p(T_k) \leq p$. We will deduce a contradiction. Let $X$ be an $\eta_p(T_k)$-set such that $|X \cap V(T_{k-1})|$ is as large as possible. Then $|X \cap V(T_{k-1})| \leq \gamma_p(T_{k-1})$ by the choice of $X$ and, by Theorem 2.3,

$$\eta_p(V(T_k), X, T_k) = r_p(T_k) \leq p. \tag{4.1}$$

In addition, we also have the following three facts.

**Fact 1** *At most one leaf of $T_k$ is not in $X$.*

Otherwise, $\eta_p(V(T_k), X, T_k) \geq 2(p-1) > p$ by (2.1) and $p \geq 3$, which contradicts with (4.1).

**Fact 2** $|X \cap V(T_{k-1})| = \gamma_p(T_{k-1}) - 1$ *or* $\gamma_p(T_{k-1})$.

Suppose, to be contrary, that $|X \cap V(T_{k-1})| \leq \gamma_p(T_{k-1}) - 2$. Since each $\eta_p(T_{k-1})$-set has cardinality $\gamma_p(T_{k-1}) - 1$ by Observation 2.2, $\eta_p(V(T_{k-1}), X \cap V(T_{k-1}), T_{k-1}) \geq \eta_p(T_{k-1}) + 1$. Note the assumption that $r_p(T_{k-1}) = p + 1$. By (2.1)$\sim$(2.3) and Theorem 2.3, we have

$$\begin{aligned} \eta_p(V(T_k), X, T_k) &\geq \eta_p(V(T_{k-1}), X, T_k) \\ &\geq \eta_p(V(T_{k-1}), X \cap V(T_{k-1}), T_{k-1}) - 1 \\ &\geq (\eta_p(T_{k-1}) + 1) - 1 \\ &= r_p(T_{k-1}) \\ &= p + 1. \end{aligned}$$

This contradicts with (4.1). Hence $|X \cap V(T_{k-1})| = \gamma_p(T_{k-1}) - 1$ or $\gamma_p(T_{k-1})$.

**Fact 3** *If* $|X \cap V(T_{k-1})| = \gamma_p(T_{k-1}) - 1$, *then* $\{x, y\} \cap X = \{x\}$ *and* $\eta_p(V(G_j), X, T_k) = 0$.

Suppose, to be contrary, that $\{x, y\} \cap X \neq \{x\}$ or $\eta_p(V(G_j), X, T_k) > 0$. From (2.1)$\sim$(2.3) and Theorem 2.3, we will deduce a contradiction with (4.1). If $\{x, y\} \cap X \neq \{x\}$, then

$$\begin{aligned} \eta_p(V(T_k), X, T_k) &\geq \eta_p(V(T_{k-1}), X, T_k) \\ &= \eta_p(V(T_{k-1}), V(T_{k-1}) \cap X, T_{k-1}) \\ &\geq \eta_p(T_{k-1}) = r_p(T_{k-1}) = p + 1, \end{aligned}$$

a contradiction with (4.1). If $\eta_p(V(G_j), X, T_k) > 0$, then

$$\begin{aligned} \eta_p(V(T_k), X, T_k) &= \eta_p(V(G_j), X, T_k) + \eta_p(V(T_{k-1}), X, T_k) \\ &> \eta_p(V(T_{k-1}), X, T_k) \\ &\geq \eta_p(V(T_{k-1}), X \cap V(T_{k-1}), T_{k-1}) - 1 \\ &\geq r_p(T_{k-1}) - 1 \\ &= p, \end{aligned}$$

which is also a contradiction with (4.1). Hence Fact 3 holds.

Based on the above facts, we complete the proof of Lemma 4.2 by distinguishing the following two cases.

**Case 1.** $j = 1$ or $2$.

Then $G_j = K_{1,\ell}$ with center $x$, where $\ell = p - 1$ if $j = 1$, otherwise $\ell = p$. Note that $X$ is an $\eta_p(T_k)$-set. By Observation 2.2, we have

$$|X| = \gamma_p(T_k) - 1 = |A(T_k)| - 1 = |A(T_{k-1})| + \ell - 1 = \gamma_p(T_{k-1}) + \ell - 1. \tag{4.2}$$

By Fact 2, we only need to consider the following two subcases.

If $|X \cap V(T_{k-1})| = \gamma_p(T_{k-1})$, then $|X \cap V(G_j)| = \ell - 1$ by (4.2), that is, exactly two vertices are not in $X$. Since $G_j = K_{1,\ell}$ is a star with center $x$, it follows from Fact 1 that $x \notin X$ and there is a unique leaf $x'$ of $G_j$ not in $X$. Thus

$$\begin{aligned} p &= \eta_p(x', X, T_k) \\ &= \eta_p(V(T_k), X, T_k) - \eta_p(V(T_{k-1}), X, T_k) - \eta_p(x, X, T_k) \\ &\leq p - \eta_p(V(T_{k-1}), X, T_k) - \eta_p(x, X, T_k), \end{aligned}$$

which implies that $\eta_p(V(T_{k-1}), X, T_k) = \eta_p(x, X, T_k) = 0$. Together with $x' \notin X$, $x \notin X$ and $\eta_p(x, X, T_k) = 0$, we have

$$|N_{T_k}(x) \setminus \{x'\}| \geq |N_{T_k}(x) \cap X| \geq p,$$

and so $\ell = p$ (this means that $j = 2$ and $T_k$ is obtained from $T_{k-1}$ by $\mathscr{O}_2$) and $y \in X$. By $y \in X$,

$$\eta_p(V(T_{k-1}), X \cap V(T_{k-1}), T_{k-1}) = \eta_p(V(T_{k-1}), X, T_k) = 0,$$

which implies that $X \cap V(T_{k-1})$ is a $p$-dominating set of $T_{k-1}$. Further, $X \cap V(T_{k-1}) = A(T_{k-1})$ since $|X \cap V(T_{k-1})| = \gamma_p(T_{k-1})$ and $A(T_{k-1})$ is a unique $\gamma_p(T_{k-1})$-set. By the condition of $\mathscr{O}_2$, we have $y \notin A(T_{k-1}) = X \cap V(T_{k-1})$, a contradiction with $y \in X$.

If $|X \cap V(T_{k-1})| = \gamma_p(T_{k-1}) - 1$, then $|X \cap V(G_j)| = \ell$ by (4.2) and $x \in X$ by Fact 3. Since $G_j = K_{1,\ell}$ is a star with center $x$, there is a leaf of $G_j$ not in $X$. Hence $\eta_p(V(G_j), X, T_k) \geq p - 1 > 0$, which contradicts with the result $\eta_p(V(G_j), X, T_k) = 0$ in Fact 3.

**Case 2.** $j = 3$ or 4.

Then $G = F_{p-1}$ or $G_{t,p-1}$ $(t \geq p)$ with center $x$. Since $X$ is an $\eta_p(T_k)$-set, it is easily seen from Observation 2.2 that

$$|X| = \gamma_p(T_k) - 1 = |A(T_k)| - 1 = \gamma_p(G_j) + |A(T_{k-1})| - 1 = \gamma_p(G_j) + \gamma_p(T_{k-1}) - 1. \quad (4.3)$$

**Subcase 2.1.** $|X \cap V(T_{k-1})| = \gamma_p(T_{k-1})$.

It is clear that $|X \cap V(G_j)| = \gamma_p(G_j) - 1$ by (4.3). Since $r_p(G_j) = p + 1$ by Lemma 4.1, we can deduce $\{x, y\} \cap X = \{y\}$ and $\eta_p(V(T_{k-1}), X, T_k) = 0$ by the similar proof of Fact 3.

If $j = 3$, then $G_j = F_{p-1}$ with center $x$ and $\mu_p(x, \mathscr{D}_{G_j}, G_j) = p+2$ by (2.4). Applying Theorem 3.3, we obtain that $\eta_p(V(G_j), X \cap V(G_j), G_j) \geq p + 2$ from $|X \cap V(G_j)| < \gamma_p(G_j)$ and $x \notin X$. By (4.1), we get a contradiction that

$$\begin{aligned} p &\geq \eta_p(V(T_k), X, T_k) = \eta_p(V(G_j), X, T_k) \\ &\geq \eta_p(V(G_j), X \cap V(G_j), G_j) - 1 \\ &\geq p + 1. \end{aligned}$$

If $j = 4$, then $G_j = F_{t,p-1}$ $(t \geq p)$ with center $x$ and, by Figure 2, every vertex of $\mathscr{D}_{G_j} - \{x\}$ has degree 1 in $G_j - x$. Thus there is at most one vertex in $\mathscr{D}_{G_j}$ but

not in $X$ (otherwise, we have $\eta_p(V(T_k), X, T_k) = \eta_p(V(G_j), X, T_k) \geq 2(p-1) > p$ by $x \notin X$ and $p \geq 3$, which contradicts with (4.1)). Combining this with $|X \cap V(G_j)| = \gamma_p(G_j) - 1 = |\mathscr{D}_{G_j}| - 1$, we have $X \cap V(G_j) \subset \mathscr{D}_{G_j}$. Let $\mathscr{D}_{G_j} - X = \{u\}$ and $v$ be the unique neighbor of $u$ in $G_j - x$. Then

$$\begin{aligned}\eta_p(V(T_k), X, T_k) &\geq \eta_p(u, X, T_k) + \eta_p(v, X, T_k)\\ &= \eta_p(u, X \cap V(G_j), G_j) + \eta_p(v, X \cap V(G_j), G_j)\\ &\geq p + 1.\end{aligned}$$

This contradicts with (4.1).

**Subcase 2.2.** $|X \cap V(T_{k-1})| = \gamma_p(T_{k-1}) - 1$.

By Fact 3, $\{x, y\} \cap X = \{x\}$ and $\eta_p(V(G_j), X, T_k) = 0$. So $X \cap V(G_j)$ is a $p$-dominating set of $G_j$, moreover, has cardinality $\gamma_p(G_j)$ by (4.3). This means that $X \cap V(G_j)$ is a $\gamma_p(G_j)$-set. So $X \cap V(G_j) = \mathscr{D}_{G_j}$ since $\mathscr{D}_{G_j}$ is a unique $\gamma_p(G_j)$-set. By $x \in X$ and Figure 2, we have $G_j = F_{p-1}$, that is, $T_k$ is obtained from $T_{k-1}$ by $\mathscr{O}_3$. Thus $y \in A(T_{k-1})$ by the condition of $\mathscr{O}_3$.

We first show that there is a neighbor $z$ of $y$ in $A(T_{k-1})$ but not in $X$. For this aim, we consider that $|N_{T_{k-1}}(y) \cap A(T_{k-1})|$ and $|N_{T_{k-1}}(y) \cap X|$.

Applying the inductive assumption on $T_{k-1}$, $A(T_{k-1})$ is a unique $\gamma_p(T_{k-1})$-set and $r_p(T_{k-1}) = p+1$. It follows from (2.4)$\sim$(2.6) and Theorem 2.4 that $\mu_p(y, A(T_{k-1}), T_{k-1}) \geq \mu_p(T_{k-1}) \geq r_p(T_{k-1}) = p + 1$. Further, we can claim that

$$\mu_p(y, A(T_{k-1}), T_{k-1}) = p + 1. \tag{4.4}$$

Assume, to the contrary, that $\mu_p(y, A(T_{k-1}), T_{k-1}) \geq p + 2$. On the one hand, by $r_p(T_{k-1}) = p + 1$ and Theorem 3.3, we have $\eta_p(V(T_{k-1}), X \cap V(T_{k-1}), T_{k-1}) \geq p + 2$, and so

$$\eta_p(V(T_{k-1}), X, T_k) \geq \eta_p(V(T_{k-1}), X \cap V(T_{k-1}), T_{k-1}) - 1 \geq p + 1.$$

On the other hand, by (4.1) and $\eta_p(V(G_j), X, T_k) = 0$, we have

$$\eta_p(V(T_{k-1}), X, T_k) = \eta_p(V(T_k), X, T_k) - \eta_p(V(G_j), X, T_k) \leq p,$$

a contradiction. Thus (4.4) holds.

For convenience, let $A_y = N_{T_{k-1}}(y) \cap A(T_{k-1})$. Together with (4.4), (2.4) and the condition of $\mathscr{O}_3$, we have

$$\begin{aligned}p + 1 &= \mu_p(y, A(T_{k-1}), T_{k-1})\\ &= |PN_p(y, A(T_{k-1}), T_{k-1})| + \max\{0, p - |A_y|\}\\ &\geq \begin{cases} (p+1) + 0 & \text{if } |A_y| \geq p;\\ (|A_y| + 2) + (p - |A_y|) & \text{if } |A_y| < p \end{cases}\\ &= \begin{cases} p + 1 & \text{if } |A_y| \geq p;\\ p + 2 & \text{if } |A_y| < p, \end{cases}\end{aligned}$$

which implies that

$$|N_{T_{k-1}}(y) \cap A(T_{k-1})| = |A_y| \geq p \text{ and } |PN_p(y, A(T_{k-1}), T_{k-1})| = p + 1. \tag{4.5}$$

Since $|X \cap V(T_{k-1})| = \gamma_p(T_{k-1}) - 1$ and $r_p(T_{k-1}) = p+1$, it follows from (2.1)$\sim$(2.3) and Theorem 2.3 that $\eta_p(V(T_{k-1}), X \cap V(T_{k-1}), T_{k-1}) \geq \eta_p(T_{k-1}) = r_p(T_{k-1}) = p + 1$. By (4.1), we have

$$|N_{T_{k-1}}(y) \cap X| < p. \tag{4.6}$$

(4.5) and (4.6) implies that there is a neighbor $z$ of $y$ in $A(T_{k-1})$ but not in $X$. Let $T_z$ be the component of $T_{k-1} - y$ containing $z$. By $z \in A(T_{k-1})$, we have $z \notin PN_p(y, A(T_{k-1}), T_{k-1})$. By **(ii)** of Theorem 3.2, $r_p(T_z) = p + 1$ and $A(T_{k-1}) \cap V(T_z)$ is a unique $\gamma_p(T_z)$-set.

If $|X \cap V(T_z)| \geq \gamma_p(T_z)$, then let $X' = (X - V(T_z)) \cup (A(T_{k-1}) \cap V(T_z))$. Clearly, $|X'| \leq |X|$. Together with $y \notin X$, $z \in A(T_{k-1}) - X$ and (4.6), we have

$$\begin{aligned}
\eta_p(V(T_k), X', T_k) &= \eta_p(V(T_k) - \{y\} - V(T_z), X', T_k) + \eta_p(y, X', T_k) + \eta_p(V(T_z), X', T_k) \\
&= \eta_p(V(T_k) - \{y\} - V(T_z), X, T_k) + (\eta_p(y, X, T_k) - 1) + 0 \\
&= \eta_p(V(T_k) - V(T_z), X, T_k) - 1 \\
&\leq \eta_p(V(T_k), X, T_k) - 1
\end{aligned}$$

which contradicts that $X$ is an $\eta_p(T_{k-1})$-set.

If $|X \cap V(T_z)| < \gamma_p(T_z)$, then, by $y \notin X$ and $r_p(T_z) = p + 1$, we have

$$\begin{aligned}
\eta_p(V(T_k), X, T_k) &\geq \eta_p(V(T_z), X, T_k) \\
&= \eta_p(V(T_z), X \cap V(T_z), T_z) \\
&\geq \eta_p(T_z) = r_p(T_z) = p + 1,
\end{aligned}$$

which contradicts with (4.1).

This completes the proof of Lemma 4.2. ∎

Let $p \geq 1$ be an integer. For a tree $T$ with a unique $\gamma_p(T)$-set $\mathscr{D}_T$, we use $\ell_p(T)$ to denote the number of all $p$-private neighbors with respect to $\mathscr{D}_T$ in $T$. Since every $p$-private neighbor with respect to $\mathscr{D}_T$ has exactly $p$ neighbors in $\mathscr{D}_T$, we have

$$\ell_p(T) = \frac{1}{p} \sum_{x \in \mathscr{D}_T} |PN_p(x, \mathscr{D}_T, T)|. \tag{4.7}$$

**Lemma 4.3** *Let $p \geq 3$ be an integer and $T$ be a tree. If $r_p(T) = p + 1$, then $T \in \mathscr{T}_p$.*

**Proof.** By Theorem 3.1, $T$ has a unique $\gamma_p(T)$-set $\mathscr{D}_T$ and each vertex in $\mathscr{D}_T$ has at least one $p$-private neighbor with respect to $\mathscr{D}_T$. By $|\mathscr{D}_T| = \gamma_p(T) > p$ and (4.7), we have

$$\ell_p(T) = \frac{1}{p} \sum_{x \in \mathscr{D}_T} |PN_p(x, \mathscr{D}_T, T)| \geq \frac{1}{p} \sum_{x \in \mathscr{D}_T} 1 = \frac{1}{p} |\mathscr{D}_T| > 1,$$

that is, $\ell_p(T) \geq 2$ since $\ell_p(T)$ is an integer. We will show $T \in \mathscr{T}_p$ by induction on $\ell_p(T)$.

If $\ell_p(T) = 2$, then let $x$ and $y$ be two $p$-private neighbors with respect to $\mathscr{D}_T$. Since every vertex in $\mathscr{D}_T$ has at least one $p$-privated neighbor with respect to $\mathscr{D}_T$, we have

$$\mathscr{D}_T = (N_T(x) \cap \mathscr{D}_T) \cup (N_T(y) \cap \mathscr{D}_T).$$

From the fact that $x$ and $y$ have at most one common neighbor in $T$, we know that

$$\gamma_p(T) = |\mathscr{D}_T| = \begin{cases} 2p-1 & \text{if } |N_T(x) \cap N_T(y)| = 1; \\ 2p & \text{if } |N_T(x) \cap N_T(y)| = 0. \end{cases}$$

Combining with $r_p(T) = p + 1$, we can check easily that

$$T = \begin{cases} F_{p-1} & \text{if } \gamma_p(T) = 2p-1; \\ S_{p,p} & \text{if } \gamma_p(T) = 2p, \end{cases}$$

where $S_{p,p}$ is obtained by attaching $p-1$ leaves at every endvertex of a path $P_2$. Clearly, $T$ can be obtained from star $T_0 = K_{1,p}$ by $\mathscr{O}_1$ if $T = F_{p-1}$, otherwise by $\mathscr{O}_2$. So $T \in \mathscr{T}_p$. This establishes the base case.

Let $\ell_p(T) \geq 3$. Assume that, for any tree $T'$ with $r_p(T') = p + 1$, if $\ell_p(T') < \ell_p(T)$ then $T' \in \mathscr{T}_p$ .

For a leaf $r$ of $T$, we root $T$ at $r$. Let $d = \max\{d_T(r, x) : x \in V(T)\}$. Thus we can partite $V(T)$ into $\{V_0, V_1, \ldots, V_d\}$, where

$$V_i = \{x \in V(T) \quad : \quad \max\{d_T(x, v) : v \in D[x]\} = i\}, \quad \text{for } i = 0, 1, \ldots, d.$$

By the definitions of $V_i$, $V_0 = L(T) - \{r\}$ and $V_d = \{r\}$. Clearly, $V_0 \subseteq \mathscr{D}_T$ by Observation 2.1 and $d \geq 3$ (otherwise, $T$ is a star, a contradiction with $\ell_p(T) \geq 3$).

We now turn to consider $V_1$. For any $x \in V_1$, we know from the definition of $V_1$ that $x$ is a stem of $T$ and $C(x) \subseteq V_0$. By $r_p(T) = p+1$, $x$ is a $p$-private neighbor with respect to $\mathscr{D}_T$, and so $deg_T(x) = p$ or $p + 1$.

**Claim.** If there exists a vertex $x \in V_1$ with $deg_T(x) = p + 1$, then $T \in \mathscr{T}_p$.

**Proof of Claim 4.** Let $T_x = T[D[x]]$ and $T' = T - D[x]$. By $deg_T(x) = p + 1$, $T_x = K_{1,p}$ with center $x$, and so $T = K_{1,p} \uplus_{xy} T'$, where $y$ is the father of $x$ in $T$.

Since $|C(x)| = deg_T(x) - 1 = p$ and $x$ is a $p$-private neighbor with respect to $\mathscr{D}_T$, we have $y \notin \mathscr{D}_T$ and $C(x)$ (resp. $\mathscr{D}_T \cap V(T')$) is a $p$-dominating set of $T_x$ (resp. $T'$). So

$$\gamma_p(T) = |\mathscr{D}_T| = |C(x)| + |\mathscr{D}_T \cap V(T')| \geq \gamma_p(T_x) + \gamma_p(T'). \tag{4.8}$$

Further, we have $\gamma_p(T) = \gamma_p(T_x) + \gamma_p(T')$ since the union between a $\gamma_p(T_x)$-set and a $\gamma_p(T')$-set is a $p$-dominating set of $T$. Hence, $\mathscr{D}_T \cap V(T')$ is a $\gamma_p(T')$-set.

Since $x \notin \mathscr{D}_T$ and $y \notin \mathscr{D}_T$, we have $\gamma_p(T') \geq p$, further, $\gamma_p(T') \geq p+1$ by $\ell_p(T) \geq 3$. Together with Theorem 1.1, (4.8) and Corollary 2.1, we have

$$p + 1 \geq r_p(T') \geq r_p(T) = p + 1,$$

which implies that $r_p(T') = p+1$. By Theorem 3.1, $V(T') \cap \mathscr{D}_T$ is a unique $\gamma_p(T')$-set. Noting that $x$ is a $p$-private neighbor with respect to $\mathscr{D}_T$, we have $\ell_p(T') = \ell_p(T) - 1$. Applying the induction on $T'$, $T' \in \mathscr{T}_p$ and $\mathscr{D}_{T'} = \mathscr{D}_T \cap V(T') = A(T')$ by Lemma 4.2. Since $T = K_{1,p} \uplus_{xy} T'$ and $y \notin \mathscr{D}_T \cap V(T') = A(T')$, $T$ is obtained from $T'$ by $\mathscr{O}_2$, and so $T \in \mathscr{T}_p$. $\Box$

In the following, by Claim 4, we only need to consider the case that

$$deg_T(v) = p, \quad \text{for each } v \in V_1. \tag{4.9}$$

Then the father of each vertex in $V_1$ belongs to $\mathscr{D}_T$, and so $V_2 \subseteq \mathscr{D}_T$.

Let $x \in V_3$ and $P = xwvu$ be a path in $T[D[x]]$ such that $deg_T(w)$ is as large as possible. Obviously, $u \in V_0$, $v \in V_1$ and $w \in V_2$. By (2.4)$\sim$(2.6) and Theorem 2.4,

$$\mu_p(w, \mathscr{D}_T, T) \geq \mu_p(T) \geq r_p(T) = p+1.$$

**Case 1.** $\mu_p(w, \mathscr{D}_T, T) \geq p+2$.

Let $T' = T - D[v]$. Since $v \in V_1$ and $|D(v)| = deg_T(v) - 1 = p-1$, $T[D[v]] = K_{1,p-1}$ with center $v$, and so $T = K_{1,p-1} \uplus_{vw} T'$.

We first show that $r_p(T') = p+1$. Due to $w \in V_2 \subseteq \mathscr{D}_T$, $\mathscr{D}_T \cap V(T')$ is a $p$-dominating set of $T'$, and so

$$\gamma_p(T) = |\mathscr{D}_T| = |\mathscr{D}_T \cap V(T')| + (p-1) \geq \gamma_p(T') + p - 1. \tag{4.10}$$

Let $X'$ be an $\eta_p(T')$-set and $X = X' \cup D(v)$. Then, by (4.10),

$$|X| = |X'| + |D(v)| = (\gamma_p(T') - 1) + (p-1) < \gamma_p(T).$$

If $w \in X'$, then $r_p(T') = \eta_p(T') = \eta_p(V(T'), X', T') = \eta_p(V(T), X, T) \geq r_p(T) = p+1$ by (2.1)$\sim$(2.3) and Theorem 2.3. If $w \notin X'$, then $\eta_p(V(T), X, T) \geq p+2$ by $\mu_p(w, \mathscr{D}_T, T) \geq p+2$ and Theorem 3.3, and so $r_p(T') = \eta_p(V(T'), X', T') = \eta_p(V(T), X, T) - 1 \geq p+1$. We know $r_p(T') = p+1$ from Theorem 1.1.

By Theorem 3.1, $r_p(T') = p+1$ implies that $T'$ has a unique $\gamma_p(T')$-set $\mathscr{D}_{T'}$. We claim that $\mathscr{D}_{T'} = \mathscr{D}_T \cap V(T')$. To be contrary, then $|\mathscr{D}_T \cap V(T')| \geq \gamma_p(T') + 1$ since $\mathscr{D}_T \cap V(T')$ is a $p$-dominating set of $T'$, and so, by (4.10),

$$|\mathscr{D}_{T'} \cup D[v]| = \gamma_p(T') + p \leq |\mathscr{D}_T \cap V(T')| - 1 + p = |\mathscr{D}_T|. \tag{4.11}$$

Since $\mathscr{D}_{T'} \cup D[v]$ is a $p$-dominating set of $T$, (4.11) implies that $\mathscr{D}_{T'} \cup D[v]$ is a $\gamma_p(T)$-set different to $\mathscr{D}_T$, a contradiction. The claim holds.

It is easily seen that $\ell_p(T') = \ell_p(T) - 1$ from $\mathscr{D}_{T'} = \mathscr{D}_T \cap V(T')$ and $v \in PN_p(w, \mathscr{D}_T, T)$. Applying the induction on $T'$, $T' \in \mathscr{T}_p$ and $V(T') \cap \mathscr{D}_T = \mathscr{D}_{T'} = A(T')$ by Lemma 4.2. Due to $w \in \mathscr{D}_T \cap V(T') = A(T')$, $T$ can be obtained from $T'$ by $\mathscr{O}_1$, and so $T \in \mathscr{T}_p$.

**Case 2.** $\mu_p(w, \mathscr{D}_T, T) = p+1$.

Since $r_p(T) = p+1$ and $w \in \mathscr{D}_T$, $w$ is not a stem of $T$, and so $C(w) \subseteq V_1$. Note that $N_T(w) = C(w) \cup \{x\}$ and $\mu_p(w, \mathscr{D}_T, T) = p+1$. We can conclude from (2.4) that either $|C(w)| = 2$ and $x \in \mathscr{D}_T$ or $|C(w)| = 1$ and $x \notin \mathscr{D}_T \cup PN_p(w, \mathscr{D}_T, T)$ . We distinguish the following two subcases.

**Subcase 2.1.** $|C(w)| = 2$ and $x \in \mathscr{D}_T$.

Let $T' = T - D[w]$. Since $C(w) \subseteq V_1$ and $|C(w) = 2|$, $T[D[w]] = F_{p-1}$ with center $w$, and so $T = F_{p-1} \uplus_{wx} T'$.

Due to $x \in \mathscr{D}_T$, $x \notin PN_p(w, \mathscr{D}_T, T)$, and so, by **(ii)** of Theorem 3.2, we have $r_p(T') = p+1$ and $\mathscr{D}_{T'} = \mathscr{D}_T \cap V(T')$. So $\ell_p(T') = \ell_p(T) - |C(w)| < \ell_p(T)$. Applying the induction on $T'$, $T' \in \mathscr{T}_p$ and, by Lemma 4.2, $A(T') = \mathscr{D}_{T'} = \mathscr{D}_T \cap V(T')$.

Since $x \in \mathscr{D}_T$ and $r_p(T) = p+1$, by (2.4)$\sim$(2.6) and Theorem 2.4, we have

$$\begin{aligned} |PN_p(x, \mathscr{D}_T, T)| + \max\{0, p - |N_T(x) \cap \mathscr{D}_T|\} &= \mu_p(x, \mathscr{D}_T, T) \\ &\geq \mu_p(T) \geq r_p(T) = p+1, \end{aligned}$$

which implies $|PN_p(x, \mathscr{D}_T, T)| \geq \min\{p+1, |N_T(x) \cap \mathscr{D}_T| + 1\}$. From $w \in N_T(x) \cap \mathscr{D}_T$ and $A(T') = \mathscr{D}_T \cap V(T')$, we know that

$$|PN_p(x, A(T'), T')| = |PN_p(x, \mathscr{D}_T, T)| \quad \text{and} \quad |N_{T'}(x) \cap A(T')| = |N_T(x) \cap \mathscr{D}_T| - 1.$$

Hence $|PN_p(x, A(T'), T')| \geq \min\{p+1, |N_{T'}(x) \cap A(T')| + 2\}$. Thus $T$ can be obtained from $T'$ by $\mathscr{O}_3$ and so $T \in \mathscr{T}_p$.

**Subcase 2.2.** $|C(w)| = 1$ and $x \notin \mathscr{D}_T \cup PN_p(w, \mathscr{D}_T, T)$.

Let $T' = T - D[x]$. By $x \in V_3$, $C(x) \subseteq V_0 \cup V_1 \cup V_2$. Note that the unique neighbor of each vertex in $V_0$ is a $p$-private neighbor with respect to $\mathscr{D}_T$ (this means that $\mathscr{D}_T \cap V_1 = \emptyset$) and, to $p$-dominate $V_1$, all neighbors of each vertex in $V_1$ must belong to $\mathscr{D}_T$ by (4.9). So we can derive from $x \notin \mathscr{D}_T \cup PN_p(w, \mathscr{D}_T, T)$ that $C(x) \subseteq V_2$ and $|C(x)| \geq p$. By the choice of $P = xwvu$ and $|C(w)| = 1$, all vertices in $C(x)$ have degree 2 in $T$. Hence $T[D[x]] = F_{t,p-1}$ with center $x$, where $t = |C(x)| \geq p$. By $deg_T(x) \geq p > 1 = deg_T(r)$, we know that $x \neq r$ and so $x \in D(r)$. Let $y$ be the father of $x$, then $T = F_{t,p-1} \uplus_{xy} T'$.

Since $x \notin \mathscr{D}_T \cup PN_p(w, \mathscr{D}_T, T)$, $\mathscr{D}_T \cap V(F_{t,p-1})$ (resp. $\mathscr{D}_T \cap V(T')$) is a $p$-dominating set of $F_{t,p-1}$ (resp. $T'$). So,

$$\gamma_p(T) = |\mathscr{D}_T| = |\mathscr{D}_T \cap V(F_{t,p-1})| + |\mathscr{D}_T \cap V(T')| \geq \gamma_p(F_{t,p-1}) + \gamma_p(T').$$

Further, $\gamma_p(T) = \gamma_p(F_{t,p-1}) + \gamma_p(T')$ since the union between a $\gamma_p(F_{t,p-1})$-set and a $\gamma_p(T')$-set is a $p$-dominating set of $T$. Hence $\mathscr{D}_T \cap V(F_{t,p-1})$ (resp. $\mathscr{D}_T \cap V(T')$) is a $\gamma_p(F_{t,p-1})$-set (resp. $\gamma_p(T')$-set).

If $\gamma_p(T') \leq p$, then we can check easily that $T'$ is a star $K_{1,p}$ from $x \notin \mathscr{D}_T \cup PN_p(w, \mathscr{D}_T, T)$ and $r_p(T) = p+1$. So $T$ can be obtained from $K_{1,p}$ by $\mathscr{O}_4$ and $T \in \mathscr{T}_p$.

If $\gamma_p(T') \geq p+1$, then, by Corollary 2.1, $p+1 = r_p(T) \leq r_p(T')$. It follows that $r_p(T') = p+1$ by Theorem 1.1, and so $\mathscr{D}_{T'} = \mathscr{D}_T \cap V(T')$ by Theorem 3.1. Note

that $\ell_p(T') = \ell_p(T) - \ell_p(F_{t,p-1}) < \ell_p(T)$. Applying the induction on $T'$, $T' \in \mathscr{T}_p$ and $A(T') = \mathscr{D}_{T'} = \mathscr{D}_T \cap V(T')$ by Lemma 4.2 and Theorem 3.1. Hence $T$ can be obtained from $T'$ by $\mathscr{O}_4$ and $T \in \mathscr{T}_p$. ∎

Lemmas 4.2 and 4.3 imply that Theorem 4.1 is true.

**Theorem 4.1** *For an integer $p \geq 3$ and a tree $T$, $r_p(T) = p+1$ if and only if $T \in \mathscr{T}_p$.*

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